Appendix G: Matrices, Determinants, and Systems of Equations

G.1 Matrix Definitions and Notations

Matrix

An $m \times n$ *matrix* is a rectangular or square array of elements with m rows and n columns. An example of a matrix is shown in Eq. (G.1).

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{G.1}$$

For each subscript, a_{ij}, $i =$ the row, and $j =$ the column. If $m = n$, the matrix is said to be a *square matrix*.

Vector

If a matrix has just one row, it is called a *row vector*. An example of a row vector follows:

$$\mathbf{B} = [\,b_{11} \quad b_{12} \quad \cdots \quad b_{1n}\,] \tag{G.2}$$

If a matrix has just one column, it is called a *column vector*. An example of a column vector follows:

$$\mathrm{C} = \begin{bmatrix} c_{11} \\ c_{12} \\ \vdots \\ c_{m1} \end{bmatrix} \tag{G.3}$$

Partitioned Matrix

A matrix can be partitioned into component matrices or vectors. For example, let

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{bmatrix}=\left[\begin{array}{cc|cc} a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ \hline a_{41} & a_{42} & a_{43} & a_{44}\end{array}\right]=\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{bmatrix} \tag{G.4}$$

where

$$\mathbf{A}_{11}=\begin{bmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\\ a_{31} & a_{32}\end{bmatrix};\quad \mathbf{A}_{12}=\begin{bmatrix} a_{13} & a_{14}\\ a_{23} & a_{24}\\ a_{33} & a_{34}\end{bmatrix}$$
$$\mathbf{A}_{21}=\begin{bmatrix} a_{41} & a_{42}\end{bmatrix};\quad \mathbf{A}_{22}=\begin{bmatrix} a_{43} & a_{44}\end{bmatrix}$$

Null Matrix

A matrix with all elements equal to zero is called the *null matrix*; that is, $a_{ij}=0$ for all i and j An example of a null matrix follows:

$$\mathbf{A}=\begin{bmatrix} 0 & 0 & 0 & & \cdots & & 0\\ 0 & 0 & 0 & & \cdots & & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix} \tag{G.5}$$

Diagonal Matrix

A square matrix with all elements off of the diagonal equal to zero is said to be a *diagonal matrix*; that is, $a_{ij}=0$ for $i\neq j$. An example of a diagonal matrix follows:

$$\mathbf{A}=\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0\\ 0 & a_{22} & 0 & \cdots & 0\\ 0 & 0 & a_{33} & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & a_{nn}\end{bmatrix} \tag{G.6}$$

Identity Matrix

A diagonal matrix with all diagonal elements equal to unity is called an *identity matrix* and is denoted by $\mathbf{I}$; that is, $a_{ij}=1$ for $i=j$, and $a_{ij}=0$ for $i\neq j$. An example of an identity matrix follows:

$$\mathbf{A}=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1\end{bmatrix} \tag{G.7}$$

Symmetric Matrix

A square matrix for which $a_{ij} = a_{ji}$ is called a *symmetric matrix*. An example of a symmetric matrix follows:

$$\mathbf{A} = \begin{bmatrix} 3 & 8 & 7 \\ 8 & 9 & 2 \\ 7 & 2 & 4 \end{bmatrix} \tag{G.8}$$

Matrix Transpose

The *transpose* of matrix $\mathbf{A}$, designated $\mathbf{A}^T$, is formed by interchanging the rows and columns of $\mathbf{A}$. Thus, if $\mathbf{A}$ is an $m \times n$ matrix with elements a_{ij}, the transpose is an $n \times m$ matrix with elements a_{ji}. An example follows. Given

$$\mathbf{A} = \begin{bmatrix} 1 & 7 & 9 \\ 2 & 6 & -3 \\ 4 & 8 & 5 \\ -1 & 3 & -2 \end{bmatrix} \tag{G.9}$$

then

$$\mathbf{A}^T = \begin{bmatrix} 1 & 2 & 4 & -1 \\ 7 & 6 & 8 & 3 \\ 9 & -3 & 5 & -2 \end{bmatrix} \tag{G.10}$$

Determinant of a Square Matrix

The *determinant* of a square matrix is denoted by det $\mathbf{A}$, or

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m1} & \cdots & a_{mn} \end{bmatrix} \tag{G.11}$$

The determinant of a 2×2 matrix,

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{G.12}$$

is evaluated as

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} \tag{G.13}$$

Minor of an Element

The *minor*, M_{ij} of element a_{ij} of det $\mathbf{A}$ is the determinant formed by removing the ith row and the jth column from det $\mathbf{A}$. As an example, consider the following determinant:

$$\det \mathbf{A} = \begin{vmatrix} 3 & 8 & 7 \\ 6 & 9 & 2 \\ 5 & 1 & 4 \end{vmatrix} \tag{G.14}$$

The minor M_{32} is the determinant formed by removing the third row and the second column from det **A**. Thus,

$$M_{32} = \begin{vmatrix} 3 & 7 \\ 6 & 2 \end{vmatrix} = -36 \tag{G.15}$$

Cofactor of an Element

The *cofactor*, C_{ij}, of element a_{ij} of det **A** is defined to be

$$C_{ij} = (-1)^{(i+j)} M_{ij} \tag{G.16}$$

For example, given the determinant of Eq. (G.14)

$$C_{21} = (-1)^{(2+1)} M_{21} = (-1)^3 \begin{vmatrix} 8 & 7 \\ 1 & 4 \end{vmatrix} = -25 \tag{G.17}$$

Evaluating the Determinant of a Square Matrix

The determinant of a square matrix can be evaluated by expanding minors along any row or column. Expanding along any row, we find

$$\det \mathbf{A} = \sum_{k=1}^{n} a_{ik} C_{ik} \tag{G.18}$$

where n = number of columns of **A**; j is the jth row selected to expand by minors; and C_{ik} is the cofactor of a_{ik}. Expanding along any column, we find

$$\det \mathbf{A} = \sum_{k=1}^{m} a_{kj} C_{kj} \tag{G.19}$$

where m = number of rows of **A**; j is the jth column selected to expand by minors; and C_{kj} is the cofactor of a_{kj}. For example, if

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 2 \\ -5 & 6 & -7 \\ 8 & 5 & 4 \end{bmatrix} \tag{G.20}$$

then, expanding by minors on the third column, we find

$$\det \mathbf{A} = 2 \begin{vmatrix} -5 & 6 \\ 8 & 5 \end{vmatrix} - (-7) \begin{vmatrix} 1 & 3 \\ 8 & 5 \end{vmatrix} + 4 \begin{vmatrix} 1 & 3 \\ -5 & 6 \end{vmatrix} = -195 \tag{G.21}$$

Expanding by minors on the second row, we find

$$\det \mathbf{A} = -(-5) \begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix} + 6 \begin{vmatrix} 1 & 2 \\ 8 & 4 \end{vmatrix} - (-7) \begin{vmatrix} 1 & 3 \\ 8 & 5 \end{vmatrix} = -195 \tag{G.22}$$

Singular Matrix

A matrix is *singular* if its determinant equals zero.

Nonsingular Matrix

A matrix is *nonsingular* if its determinant does not equal zero.

Adjoint of a Matrix

The *adjoint* of a square matrix, **A**, written adj **A**, is the matrix formed from the transpose of the matrix **A** after all elements have been replaced by their cofactors. Thus,

$$\operatorname{adj}\mathbf{A} = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^T \tag{G.23}$$

For example, consider the following matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 4 & 5 \\ 6 & 8 & 7 \end{bmatrix} \tag{G.24}$$

Hence,

$$\operatorname{adj}\mathbf{A} = \begin{bmatrix} \begin{vmatrix} 4 & 5 \\ 8 & 7 \end{vmatrix} & -\begin{vmatrix} -1 & 5 \\ 6 & 7 \end{vmatrix} & \begin{vmatrix} -1 & 4 \\ 6 & 8 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 8 & 7 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 6 & 7 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 6 & 8 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ -1 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} \end{bmatrix}^T = \begin{bmatrix} -12 & 10 & -2 \\ 37 & -11 & -8 \\ -32 & 4 & 6 \end{bmatrix} \tag{G.25}$$

Rank of a Matrix

The *rank* of a matrix, **A**, equals the number of linearly independent rows or columns. The rank can be found by finding the highest-order square submatrix that is nonsingular. For example, consider the following:

$$\mathbf{A} = \begin{bmatrix} 1 & -5 & 2 \\ 4 & 7 & -5 \\ -3 & 15 & -6 \end{bmatrix} \tag{G.26}$$

The determinant of $\mathbf{A} = 0$. Since the determinant is zero, the 3×3 matrix is singular. Choosing the submatrix

$$\mathbf{A} = \begin{bmatrix} 1 & -5 \\ 4 & 7 \end{bmatrix} \tag{G.27}$$

whose determinant equals 27, we conclude that **A** is of rank 2.

G.2 Matrix Operations

Addition

The sum of two matrices, written $\mathbf{A} + \mathbf{B} = \mathbf{C}$, is defined by $a_{ij} + b_{ij} = c_{ij}$. For example,

$$\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 7 & -5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 9 & -6 \\ -1 & 8 \end{bmatrix} \tag{G.28}$$

Subtraction

The difference between two matrices, written $\mathbf{A} - \mathbf{B} = \mathbf{C}$, is defined by $a_{ij} - b_{ij} = c_{ij}$. For example,

$$\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} 7 & -5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ 7 & 2 \end{bmatrix} \tag{G.29}$$

Multiplication

The product of two matrices, written $\mathbf{AB} = \mathbf{C}$, is defined by $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$. For example, if

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \tag{G.30}$$

then

$$\mathbf{C} = \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}) & (a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}) & (a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33}) \\ (a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}) & (a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32}) & (a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33}) \end{bmatrix} \tag{G.31}$$

Notice that muitiplicalion is defined only if the number of columns of **A** equals the number of rows of **B**.

Multiplication by a Constant

A matrix can be multiplied by a constant by multiplying every element of the matrix by that constant. For example, if

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{G.32}$$

then

$$k\mathbf{A} = \begin{bmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{bmatrix} \tag{G.33}$$

Inverse

An $n \times n$ square matrix, **A**, has an inverse, denoted by $\mathbf{A}^{-1}$, which is defined by

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} \tag{G.34}$$

where **I** is an $n \times n$ identity matrix. The inverse of **A** is given by

$$\mathbf{A}^{-1} = \frac{\text{adj } \mathbf{A}}{\det \mathbf{A}} \tag{G.35}$$

For example, find the inverse of **A** in Eq.(G.24). The adjoint was calculated in Eq. (G.25). The determinant of **A** is

$$\det \mathbf{A} = 1\begin{vmatrix} 4 & 5 \\ 8 & 7 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 3 \\ 8 & 7 \end{vmatrix} + 6\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -34 \tag{G.36}$$

Hence,

$$\mathbf{A}^{-1} = \frac{\begin{bmatrix} -12 & 10 & -2 \\ 37 & -11 & -8 \\ -32 & 4 & 6 \end{bmatrix}}{-34} = \begin{bmatrix} 0.353 & -0.294 & 0.059 \\ -1.088 & 0.324 & 0.235 \\ 0.941 & -0.118 & -0.176 \end{bmatrix} \tag{G.37}$$

G.3 Matrix and Determinant Identities

The following are identities that apply to matrices and determinants.

Matrix Identities

Commutative Law

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \tag{G.38}$$

$$\mathbf{AB} \neq \mathbf{BA} \tag{G.39}$$

Associative Law

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C} \tag{G.40}$$

$$\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C} \tag{G.41}$$

Transpose of Sum

$$(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T \tag{G.42}$$

Transpose of Product

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T \tag{G.43}$$

Determinant Identities

Multiplication of a Single Row or Single Column of a Matrix, A, by a Constant

If a single row or single column of a matrix, $\mathbf{A}$, is multiplied by a constant, k, forming the matrix, $\tilde{\mathbf{A}}$, then

$$\det \tilde{\mathbf{A}} = k \det \mathbf{A} \tag{G.44}$$

Multiplication of All Elements of an $n \times n$ Matrix, A, by a Constant

$$\det(k\mathbf{A}) = k^n \det \mathbf{A} \tag{G.45}$$

Transpose

$$\det \mathbf{A}^T = \det \mathbf{A} \tag{G.46}$$

Determinant of the Product of Square Matrices

$$\det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B} \tag{G.47}$$

$$\det \mathbf{AB} = \det \mathbf{BA} \tag{G.48}$$

G.4 Systems of Equations

Representation

Assume the following system of n linear equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n} &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n} &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn} &= b_n \end{aligned} \tag{G.49}$$

This system of equations can be represented in vector-matrix form as

$$\mathbf{Ax} = \mathbf{B} \tag{G.50}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}; \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

For example, the following system of equations,

$$5x_1 + 7x_2 = 3 \tag{G.51a}$$

$$-8x_1 + 4x_2 = -9 \tag{G.51b}$$

can be represented in vector-matrix form as $\mathbf{Ax} = \mathbf{B}$, or

$$\begin{bmatrix} 5 & 7 \\ -8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \end{bmatrix} \tag{G.52}$$

Solution via Matrix Inverse

If $\mathbf{A}$ is nonsingular, we can premultiply Eq. (G.50) by $\mathbf{A}^{-1}$, yielding the solution $\mathbf{x}$. Thus,

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{B} \tag{G.53}$$

For example, premultiplying both sides of Eq. (G.52) by $\mathbf{A}^{-1}$, where

$$\mathbf{A}^{-1} = \begin{bmatrix} 5 & 7 \\ -8 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 0.0526 & -0.0921 \\ 0.1053 & 0.0658 \end{bmatrix} \tag{G.54}$$

we solve for $\mathbf{x} = \mathbf{A}^{-1}\mathbf{B}$ as follows:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.0526 & -0.0921 \\ 0.1053 & 0.0658 \end{bmatrix} \begin{bmatrix} 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 0.987 \\ -0.276 \end{bmatrix} \tag{G.55}$$

Solution via Cramer's Rule

Equation (G.53) allows us to solve for all unknowns, x_i, where $i = 1$ to n. If we are interested in a single unknown, x_k, then Cramer's rule can be used. Given Eq. (G.50), Cramer's rule states that

$$x_k = \frac{\det \mathbf{A}_k}{\det \mathbf{A}} \tag{G.56}$$

where $\mathbf{A}_k$; is a matrix formed by replacing the kth column of $\mathbf{A}$ by $\mathbf{B}$. For example, solve Eq. (G.52). Using Eq. (G.56) with

$$\mathbf{A} = \begin{bmatrix} 5 & 7 \\ -8 & 4 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 3 \\ -9 \end{bmatrix}$$

we find

$$x_1 = \frac{\begin{vmatrix} 3 & 7 \\ -9 & 4 \end{vmatrix}}{\begin{vmatrix} 5 & 7 \\ -8 & 4 \end{vmatrix}} = \frac{75}{76} = 0.987 \tag{G.57}$$

and

$$x_2 = \frac{\begin{vmatrix} 5 & 3 \\ -8 & -9 \end{vmatrix}}{\begin{vmatrix} 5 & 7 \\ -8 & 4 \end{vmatrix}} = \frac{-21}{76} = -2.276 \tag{G.58}$$

Bibliography

Dorf, R. C. *Matrix Algebra—A Programmed Introduction*. Wiley, New York, 1969.

Kreyszig, E. *Advanced Engineering Mathematics*. 4th ed. Wiley, New York, 1979.

Wylie, C. R., Jr. *Advanced Engineering Mathematics*. 5th ed. McGraw-Hill, New York, 1982.

Founded in 1807, John Wiley & Sons, Inc. has been a valued source of knowledge and understanding for more than 200 years, helping people around the world meet their needs and fulfill their aspirations. Our company is built on a foundation of principles that include responsibility to the communities we serve and where we live and work. In 2008, we launched a Corporate Citizenship Initiative, a global effort to address the environmental, social, economic, and ethical challenges we face in our business. Among the issues we are addressing are carbon impact, paper specifications and procurement, ethical conduct within our business and among our vendors, and community and charitable support. For more information, please visit our website: www.wiley.com/go/citizenship.

ISBN 13 978-0470-54756-4

Printed in the United States of America.

10 9 8 7 6 5 4 3 2 1